Metastable versus Deterministic: Time of extinction of an Infinite System of Spiking Neurons

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Neuromat young researchers workshop

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Description of the model

The model we are interested in is composed of:

- A countable set I representing the *neurons*.
- For each neuron $i \in I$, a set $\mathbb{V}_i \subset I$ of *presynaptic neurons*.
- For each $i \in I$, two point processes $\left(N_i^*(t)\right)_{t \geq 0}$ and $\left(N_i^\dagger(t)\right)_{t \geq 0}$ representing *spiking times* and *total leak times* respectively.
- For each $i \in I$, a real-valued process $\left(X_i(t)\right)_{t \geq 0}$ representing the *membrane potential* of neuron i.

Spiking times, leaking times and membrane potential

The point process $\left(N_i^{\dagger}(t)\right)_{t\geq 0}$ is a Poisson process of some rate $\gamma \geq 0$.

The point process $\left(N_i^{*}(t)\right)_{t\geq 0}$ is characterized by the property that for any $s \leq t$

$$\mathbb{E}\Big(N_i^*(t) - N_i^*(s) | \mathscr{F}_s\Big) = \int_s^t \mathbb{E}\Big(\phi_i(X_i(u)) | \mathscr{F}_s\Big) du,$$

Where

$$X_i(t) = \sum_{j \in \mathbb{V}_i} \int_{]L_i(t), t[} dN_j^*(s),$$

and $L_i(t) = \sup\Big\{s \leq t : N_i^*(\{s\}) + N_i^{\dagger}(\{s\}) = 1\Big\}.$

$(\phi_i)_{i \in I}$ is a collection of rate functions.

One dimensional lattice with hard threshold

Spiking times, leaking times and membrane potential

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Where

$$X_i(t) = \int_{]L_i(t),t[} dN_{i-1}^*(s) + \int_{]L_i(t),t[} dN_{i+1}^*(s),$$

and $L_i(t) = \sup\Big\{s \leq t : N_i^*(\{s\}) + N_i^\dagger(\{s\}) = 1\Big\}.$

Active and quiescent

The choice of $\mathbb{1}_{x>0}$ as the rate function has the important consequence that any given neuron can essentially be in only two states at any given time.

- When $X_i(t) = 0$ (or equivalently when $N_i^*(t)$ has rate 0), we say that neuron i is *quiescent*.
- When $X_i(t) \geq 1$ (or equivalently when $N_i^*(t)$ has rate 1), we say that neuron i is *active*.

Phase transition

Theorem (P. Ferrari et al.)

Suppose that for any $i \in \mathbb{Z}$ we have $X_i(0) \geq 1$. There exists a critical value γ_c for the parameter γ, with $0 < \gamma_c < \infty$, such that for any $i \in \mathbb{Z}$

$$\mathbb{P}\Big(N_i^*([0,\infty[) \ < \infty\Big) = 1 \quad \text{if } \gamma > \gamma_c,$$

and

$$\mathbb{P}\Big(N_i^*([0,\infty[) \ = \infty\Big) > 0 \quad \text{if } \gamma < \gamma_c.$$

Sub-critical regime: A result of metastability

Result of Metastability

For some $N \in \mathbb{Z}^+$ we set $I_N = \mathbb{Z} \cap [-N, N]$, and we write $\left(X_i^N(t)\right)_{t \geq 0}$ for the membrane potential of neuron i in the version of the model defined on the finite set I_N.

We define the *extinction time* of this model

$$\sigma_N = \inf\left\{t \geq 0 : X_i^N(t) = 0 \text{ for all } i \in \mathbb{Z} \cap [-N, N]\right\}.$$

Theorem (Metastability)

If $\gamma < \gamma_c$, then we have the following convergence

$$\frac{\sigma_N}{\mathbb{E}(\sigma_N)} \xrightarrow[N \to \infty]{\mathscr{L}} \mathscr{E}(1).$$

The spiking rate process

We consider an important auxiliary process, namely the *spiking rates process*. Denoted $\big(\xi(t)\big)_{t\geq 0}$ and defined as follows

$$\forall t \geq 0,\ \forall i \in \mathbb{Z}, \qquad \xi_i(t) = \mathbb{1}_{X_i(t)>0}.$$

This process is an *interacting particle system*. It is a continuous time Markov process taking value in $\{0,1\}^{\mathbb{Z}}$. Each possible state is a doubly infinite sequence of 0 and 1, indicating in which state (quiescent or active) each neuron is.

The infinitesimal generator

The generator of the process $\big(\xi(t)\big)_{t\geq 0}$ is given by

$$\mathscr{L}f(\eta) = \gamma \sum_{i\in\mathbb{Z}} \Big(f(\pi_i^{\dagger}(\eta)) - f(\eta)\Big) + \sum_{i\in\mathbb{Z}} \eta_i \Big(f(\pi_i(\eta)) - f(\eta)\Big),$$

where the maps are given by

$$\pi_i^{\dagger}(\eta)_j = \begin{cases} 0 & \text{if } j = i, \\ \eta_j & \text{otherwise,} \end{cases}$$

and

$$\pi_i(\eta)_j = \begin{cases} 0 & \text{if } j = i, \\ \max(\eta_i, \eta_j) & \text{if } j \in \{i-1, i+1\}, \\ \eta_j & \text{otherwise.} \end{cases}$$

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We define the *extinction time* of the spiking rate process

$$\tau_N = \inf\left\{t \geq 0 : \xi_N(t)_i = 0 \text{ for all } i \in \mathbb{Z} \cap [-N, N]\right\}.$$

Theorem (Metastability)

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Theorem (Metastability)

If $\gamma < \gamma_c'$, then we have the following convergence

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N \to \infty]{\mathscr{L}} \mathscr{E}(1).$$

Super-critical regime: An asymptotically deterministic time of extinction

The result

Theorem

Suppose that $\gamma > \gamma_c$. Then the following convergence holds

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N\to\infty]{\mathbb{P}} 1.$$

The result

Theorem

Suppose that $\gamma > 1$. Then the following convergence holds

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N\to\infty]{\mathbb{P}} 1.$$

How we prove it

Proposition

Suppose that $\gamma > 1$. *Then there exists a constant* $0 < C < \infty$ *depending on* γ *such that the following convergence holds*

$$\frac{\tau_N}{\log(2N+1)} \xrightarrow[N\to\infty]{\mathbb{P}} C.$$

Proposition

Suppose that $\gamma > 1$. *Then the following convergence holds*

$$\frac{\mathbb{E}\left(\tau_N\right)}{\log(2N+1)} \xrightarrow[N\to\infty]{} C,$$

where C *is the same constant as in the previous proposition.*

Simulations

d-dimensional lattices and various activation functions.

Interaction structure

For the graph of the network we consider the lattices $\mathbb{Z}^1$, $\mathbb{Z}^2$ and $\mathbb{Z}^3$. For any $d \in \{1, 2, 3\}$, let $\|\cdot\|$ be the norm on $\mathbb{Z}^d$ given for any $j \in \mathbb{Z}^d$ by

$$\|j\| = \sum_{k=1}^{d} |j_k|,$$

where j_k is the k-th coordinate of j. The structure of the network is then given by $I = \mathbb{Z}^d$ and $\mathbb{V}_i = \{j \in I^d : \|i - j\| = 1\}$ for $i \in I$.

Activation functions

The hard threshold $\phi(x) = \mathbb{1}_{x>0}$.

The linear function $\phi(x) = x$.

The sigmoid function

$$\phi(x) = \begin{cases} (1 + e^{-3x+6})^{-1} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Simulations for the hard threshold

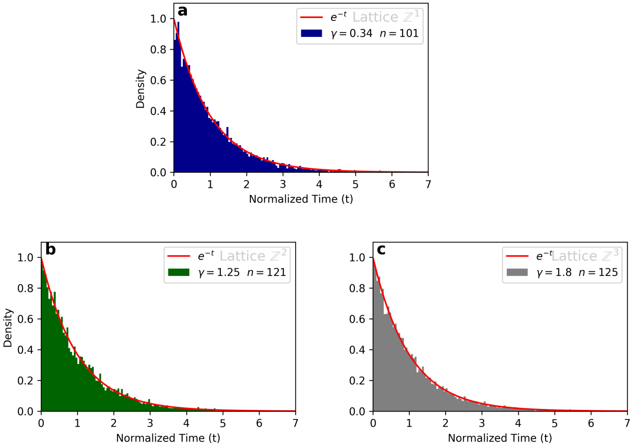

Figure: Sub-critical simulation for dimension 1, 2 and 3.

Simulations for the linear function

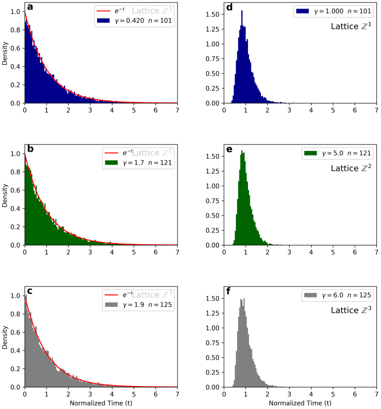

Figure: Simulations for a linear activation function in the lattices of dimension 1, 2 and 3.

Simulations for the sigmoid function

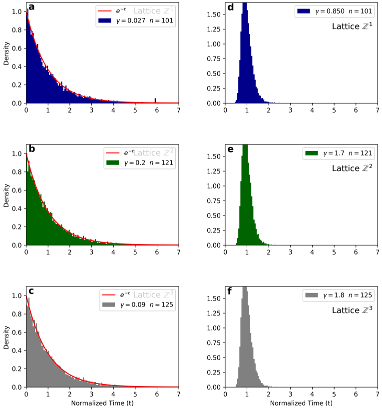

Figure: Simulations for a sigmoid activation function in the lattices of dimension 1, 2 and 3.

Simulations for a varying number of neurons in super-critical regime

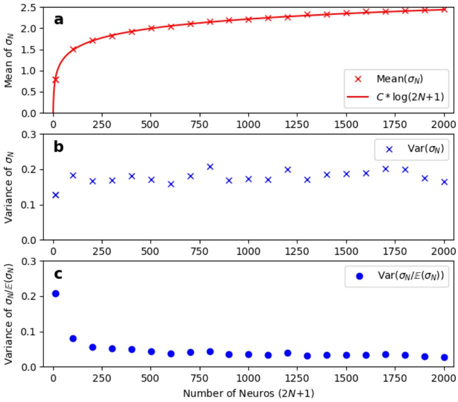

Figure: Simulations for a hard-threshold function in the lattice of dimension 1.

THE END

Thanks!