NeuroMat

Time averages of a metastable system of spiking neurons

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Overview

1. **Metastability**

2. **The Model**

3. **Result and proof**

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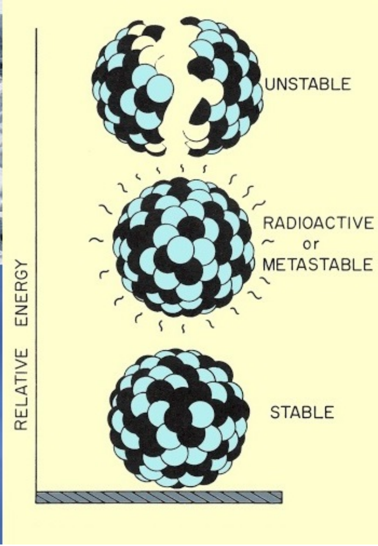
Metastability

What is Metastability in Physics (some nice pictures)



What is Metastability in statistical physics

We take as a paradigm the characterization of metastability given in "Metastable behavior of stochastic dynamics: A pathwise approach" by M. Cassandro, A. Galves, E. Olivieri and M. E. Vares (1984).

In this paradigm metastability can be described as follows. We have a stochastic process with a unique stationary measure, but if the initial conditions are suitably chosen, then

- the system stays out of equilibrium for a long and unpredictable time,
- before reaching the actual equilibrium, the system is in a regime which resemble stationarity.

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The Model

The Model and the Results

- A countable set S representing the **neurons**.
- For each neuron $i \in S$, a set $\mathbb{V}_i \subset I$ of **presynaptic neurons**.
- For each $i \in S$, two point processes $\left(N_i^*(t)\right)_{t\geq 0}$ and $\left(N_i^\dagger(t)\right)_{t\geq 0}$ representing **spiking times** and **total leak times** respectively.
- For each $i \in S$, a process $\left(X_i(t)\right)_{t\geq 0}$ taking value in $\mathbb{N}$ representing the **membrane potential** of neuron i.
- A **spiking rate function** ϕ on $\mathbb{N}$.

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The Model

The point process $\left(N_i^{\dagger}(t)\right)_{t\geq 0}$ is a Poisson process of some rate $\gamma \geq 0$.

The point process $\left(N_i^{*}(t)\right)_{t\geq 0}$ has a fluctuating rate, given at time t by $\phi\big(X_i(t)\big)$.

The membrane potential at time t for neuron i is given by

$$X_i(t) = \sum_{j\in\mathbb{V}_i} \int_{]L_i(t),t[} dN_j^*(s),$$

where $L_i(t) = \sup\left\{s \leq t : N_i^*(\{s\}) + N_i^{\dagger}(\{s\}) = 1\right\}$.

(An instantiation of) The Model

- $S = \mathbb{Z}$ or $S = \{-n, \dots, n\}$,
- $\mathbb{V}_i = \{i-1, i+1\}$ for all $i \in S$,
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Auxiliary IPS

We define the auxiliary process, denoted $\left(\xi(t)\right)_{t\geq 0}$, as follows

$$\forall t \geq 0,\ \forall i \in S, \quad \xi_i(t) \stackrel{\text{def}}{=} \mathbb{1}_{X_i(t)>0}.$$

This process is an **interacting particle system** (IPS). It is a continuous time Markov process taking value in $\{0,1\}^S$.

Depending on whether $\xi_i(t)$ is equal to 1 or 0 we will say that neuron i is **active** or **quiescent** respectively.

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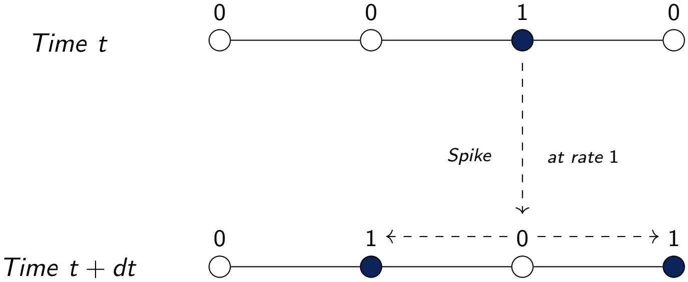
One-dimensional lattice with nearest-neighbours interaction



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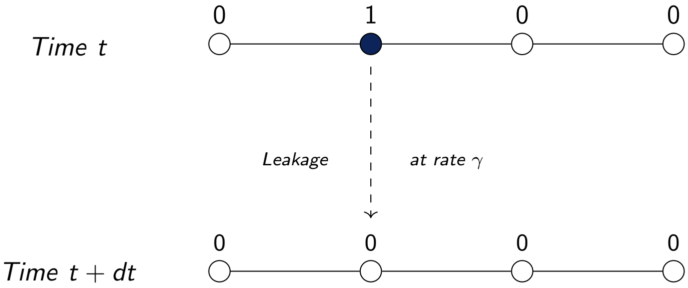
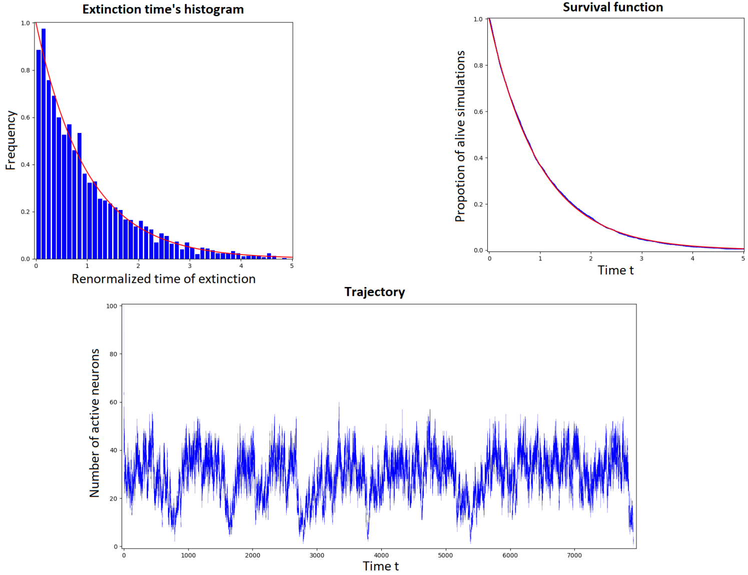


Simulations on the lattice for small γ.



Previous results (Phase transition)

Theorem (Ferrari et al. (2018))

Assuming that $X_i(0) \geq 1$ *for all* $i \in \mathbb{Z}$*, there exists some* γ_c *(satisfying* $0 < \gamma_c < \infty$*) such that the following holds*

$$\mathbb{P}\left(N_i([0,+\infty[) < \infty\right) = 1 \text{ for all } i \in \mathbb{Z} \text{ if } \gamma > \gamma_c,$$

and

$$\mathbb{P}\left(N_i([0,+\infty[) < \infty\right) < 1 \text{ for all } i \in \mathbb{Z} \text{ if } \gamma < \gamma_c.$$

Previous results (first part of metastability)

Suppose that instead of $S = \mathbb{Z}$ we take $S = [\![-n, n]\!]$ for some $n \in \mathbb{N}$, and define the **instant of the last spike**

$$T_n = \inf\{t \geq 0 : N_i^*([t, \infty[) = 0 \text{ for all } i \in S\}.$$

Theorem (M. André (2019))

If $\gamma < \gamma_c$ then the following holds

$$\frac{T_N}{\mathbb{E}(T_N)} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{E}(1).$$

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Main result (second part of metastability)

Then let $F \subset \mathbb{Z}$ be a subset of neurons satisfying $|F| < \infty$.

Theorem (Main theorem)

Suppose $0 < \gamma < \gamma_c$ and let $(R_n)_{n \geq 0}$ be an increasing sequence of positive real numbers satisfying

$$R_n \underset{n \to \infty}{\longrightarrow} +\infty \quad \text{and} \quad \frac{R_n}{\mathbb{E}\left(T_n\right)} \underset{n \to \infty}{\longrightarrow} 0.$$

There exists some $0 < \rho < 1$ (which depends only on γ) such that for any $t \geq 0$

$$\frac{1}{R_n} \sum_{i \in F} N_i\left([t, t + R_n]\right) \xrightarrow[n \to \infty]{\mathbb{P}} |F| \cdot \rho.$$

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Sketch of the proof

Fix some $\epsilon > 0$. We aim to prove

$$\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t, t+R_n]\right) - |F|\cdot\rho\right| > \epsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Here ρ actually corresponds to the asymptotic **density of active neurons** at a single site in the interacting particle system.

The main idea is to decompose this probability into two parts:

- the difference between the system of point processes and the IPS,
- and the difference between the IPS and the density of neurons.

Sketch of the proof

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\begin{aligned}
&\mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F} N_i\left([t, t+R_n]\right) - |F|\cdot\rho\right| > \epsilon\right)\\
&\leq \mathbb{P}\left(\left|\frac{1}{R_n}\sum_{i\in F}\int_t^{t+R_n} \mathbb{1}_{X_i(s)>0}\,ds - |F|\cdot\rho\right| > \frac{\epsilon}{2}\right)\\
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Further questions

- Results of metastability were obtained for this model for the lattice with nearest neighbours interaction, and for the complete interaction. Recently a result of phase transition has been obtained for regular trees (by A.M.B. Nascimento).

- Remains the question of random graphs. The most natural graph to consider is may be the Erdős–Rényi model, which has been shown to be locally like a tree in the super-critical phase.

- Can we go beyond the binary case?

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