Metastability for an Infinite System of Spiking Neurons
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Third EPEI
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Description of the model

The model we are interested in is composed of:

- A countable set I representing the *neurons*.
- For each neuron $i \in I$, a set $\mathbb{V}_i \subset I$ of *presynaptic neurons*.
- For each $i \in I$, two point processes $\left(N_i^*(t)\right)_{t\geq 0}$ and $\left(N_i^\dagger(t)\right)_{t\geq 0}$ representing *spiking times* and *total leak times* respectively.
- For each $i \in I$, a real-valued process $\left(X_i(t)\right)_{t\geq 0}$ representing the *membrane potential* of neuron i.

Spiking times, leaking times and membrane potential

The point process $\left(N_i^{\dagger}(t)\right)_{t\geq 0}$ is a Poisson process of some rate $\gamma \geq 0$.

The point process $\left(N_i^{*}(t)\right)_{t\geq 0}$ is characterized by the property that for any $s \leq t$

$$\mathbb{E}\Big(N_i^*(t) - N_i^*(s)|\mathscr{F}_s\Big) = \int_s^t \mathbb{E}\Big(\phi_i(X_i(u))|\mathscr{F}_s\Big)du,$$

Where

$$X_i(t) = \sum_{j\in\mathbb{V}_i} \int_{]L_i(t),t[} dN_j^*(s),$$

and $L_i(t) = \sup\Big\{s \leq t : N_i^*(\{s\}) + N_i^{\dagger}(\{s\}) = 1\Big\}$.

$(\phi_i)_{i\in I}$ is a collection of rate functions.

Sub-Critical Metastability

One dimensional lattice with nearest neighbour interaction and hard threshold

(the material presented in this part has been submitted to the journal of statistical physics)

Spiking times, leaking times and membrane potential

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$$\mathbb{E}\Big(N_i^*(t) - N_i^*(s)|\mathscr{F}_s\Big) = \int_s^t \mathbb{E}\Big(\ \mathbb{1}_{X_i(u)>0}\ |\mathscr{F}_s)\Big)du,$$

Where

$$X_i(t) = \sum_{j\in\mathbb{V}_i} \int_{]L_i(t),t[} dN_j^*(s),$$

and $L_i(t) = \sup\left\{s \leq t : N_i^*(\{s\}) + N_i^{\dagger}(\{s\}) = 1\right\}.$

Spiking times, leaking times and membrane potential

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Where

$$X_i(t) = \int_{]L_i(t),t[} dN_{i-1}^*(s) + \int_{]L_i(t),t[} dN_{i+1}^*(s),$$

and $L_i(t) = \sup\Big\{s \leq t : N_i^*(\{s\}) + N_i^\dagger(\{s\}) = 1\Big\}.$

Active and quiescent

The choice of $\mathbb{1}_{x>0}$ as the rate function has the important consequence that any given neuron can essentially be in only two states at any given time.

- When $X_i(t) = 0$ (or equivalently when $N_i^*(t)$ has rate 0), we say that neuron i is *quiescent*.
- When $X_i(t) \geq 1$ (or equivalently when $N_i^*(t)$ has rate 1), we say that neuron i is *active*.

Phase transition

Theorem (P. Ferrari et al.)

Suppose that for any $i \in \mathbb{Z}$ we have $X_i(0) \geq 1$. There exists a critical value γ_c for the parameter γ, with $0 < \gamma_c < \infty$, such that for any $i \in \mathbb{Z}$

$$\mathbb{P}\Big(N_i^*([0,\infty[) \ < \infty\Big) = 1 \quad \text{if } \gamma > \gamma_c,$$

and

$$\mathbb{P}\Big(N_i^*([0,\infty[) \ = \infty\Big) > 0 \quad \text{if } \gamma < \gamma_c.$$

Result of Metastability

For some $N \in \mathbb{Z}^+$ we set $I_N = \mathbb{Z} \cap [-N, N]$, and we write $\left(X_i^N(t)\right)_{t \geq 0}$ for the membrane potential of neuron i in the version of the model defined on the finite set I_N.

We define the *extinction time* of this model

$$\sigma_N = \inf \left\{ t \geq 0 : X_i^N(t) = 0 \text{ for all } i \in \mathbb{Z} \cap [-N, N] \right\}.$$

Theorem (Metastability)

If $\gamma < \gamma_c$, then we have the following convergence

$$\frac{\sigma_N}{\mathbb{E}(\sigma_N)} \xrightarrow[N \to \infty]{\mathscr{L}} \mathscr{E}(1).$$

The spiking rate process

We consider an important auxiliary process, namely the *spiking rates process*. Denoted $\left(\xi(t)\right)_{t\geq 0}$ and defined as follows

$$\forall t \geq 0,\ \forall i \in \mathbb{Z}, \quad \xi_i(t) = \mathbb{1}_{X_i(t)>0}.$$

This process is an *interacting particle system*. It is a continuous time Markov process taking value in $\{0,1\}^{\mathbb{Z}}$. Each possible state is a doubly infinite sequence of 0 and 1, indicating in which state (quiescent or active) each neuron is.

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Result of Metastability

For some $N \in \mathbb{Z}^+$ we set $I_N = \mathbb{Z} \cap [-N, N]$, and we write $(\xi_N(t))_{t \geq 0}$ for the spiking rate process restricted to the finite set I_N.

We define the *extinction time* of this model

$$\sigma_N = \inf \left\{ t \geq 0 : X_i^N(t) = 0 \text{ for all } i \in \mathbb{Z} \cap [-N, N] \right\}.$$

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For some $N \in \mathbb{Z}^+$ we set $I_N = \mathbb{Z} \cap [-N, N]$, and we write $(\xi_N(t))_{t \geq 0}$ for the spiking rate process restricted to the finite set I_N.

We define the *extinction time* of the spiking rate process

$$\tau_N = \inf\left\{t \geq 0 : \xi_N(t)_i = 0 \text{ for all } i \in \mathbb{Z} \cap [-N, N]\right\}.$$

Theorem (Metastability)

If $\gamma < \gamma_c$, then we have the following convergence

$$\frac{\sigma_N}{\mathbb{E}(\sigma_N)} \xrightarrow[N \to \infty]{\mathscr{L}} \mathscr{E}(1).$$

Result of Metastability

For some $N \in \mathbb{Z}^+$ we set $I_N = \mathbb{Z} \cap [-N, N]$, and we write $\left(\xi_N(t)\right)_{t \geq 0}$ for the spiking rate process restricted to the finite set I_N.

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Theorem (Metastability)

If $\gamma < \gamma_c$, then we have the following convergence

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N \to \infty]{\mathscr{L}} \mathscr{E}(1).$$

The infinitesimal generator

The generator of the process $\big(\xi(t)\big)_{t\geq 0}$ is given by

$$\mathscr{L}f(\eta) = \gamma \sum_{i\in\mathbb{Z}} \Big(f(\pi_i^\dagger(\eta)) - f(\eta)\Big) + \sum_{i\in\mathbb{Z}} \eta_i \Big(f(\pi_i(\eta)) - f(\eta)\Big),$$

where the maps are given by

$$\pi_i^\dagger(\eta)_j = \begin{cases} 0 & \text{if } j = i, \\ \eta_j & \text{otherwise,} \end{cases}$$

and

$$\pi_i(\eta)_j = \begin{cases} 0 & \text{if } j = i, \\ \max(\eta_i, \eta_j) & \text{if } j \in \{i-1, i+1\}, \\ \eta_j & \text{otherwise.} \end{cases}$$

The graphical representation

We can give a graphical construction of the process. For any $i \in \mathbb{N}$ let $(N_i(t))_{t\geq 0}$ and $(N_i^\dagger(t))_{t\geq 0}$ be two independent homogeneous Poisson processes with intensity 1 and γ respectively.

Let $(T_{i,n})_{n\geq 0}$ and $(T_{i,n}^\dagger)_{n\geq 0}$ be their respective jump times.

Consider the time-space diagram $\mathbb{Z} \times \mathbb{R}_+$, and do the following:

- Put a "δ" mark at the point $(i, T_{i,n}^\dagger)$,
- Put an arrow pointing from $(i, T_{i,n})$ to $(i+1, T_{i,n})$ and another pointing from $(i, T_{i,n})$ to $(i-1, T_{i,n})$.

The graphical representation

We say that there is a *valid path* from (i, t) to (j, t') if there is a chain of time segment and arrows leading from (i, t) to (j, t') such that:

- it never cross a "δ" mark,
- when moving upward, we never cross the basis of an arrow.

Then the process $(\xi(t))_{t\geq 0}$ can be defined as follows

$$\xi^A(t) = \{j \in \mathbb{Z} : (i, 0) \longrightarrow (j, t) \text{ for some } i \in A\}$$

The graphical representation

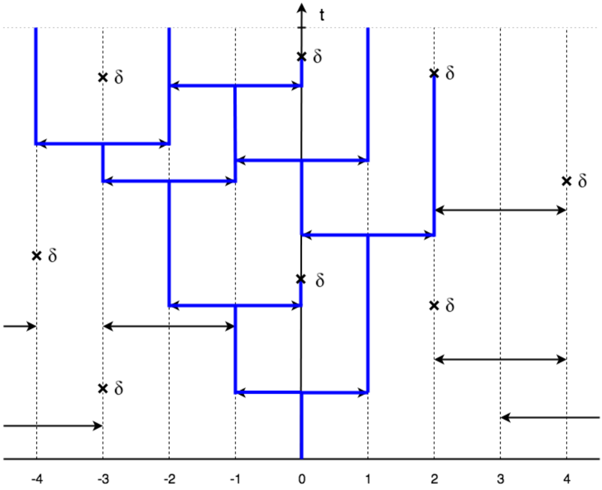

Figure: In blue all possible valid paths starting from $(0, 0)$ up to time t.

The dual process

The process $(\xi(t))_{t\geq 0}$ is *additive* in the sense that the maps satisfy the following

$$\pi_i(A) = \bigcup_{j\in A} \pi_i(\{j\}) \quad \text{and} \quad \pi_i^\dagger(A) = \bigcup_{j\in A} \pi_i^\dagger(\{j\})$$

As a consequence it has a dual process, a pure jump Markov process on $\mathscr{P}_f(\mathbb{Z})$ (finite subsets of $\mathbb{Z}$) denoted $(C(t))_{t\geq 0}$, which generator is given by

$$\tilde{\mathscr{L}} g(F) = \gamma \sum_{i\in F} \Big(g(\tilde{\pi}_i^\dagger(F)) - g(F) \Big) + \sum_{i\in F} \eta_i \Big(g(\tilde{\pi}_i(F)) - g(F) \Big),$$

The dual maps

For $F \in \mathscr{P}_f(\mathbb{Z})$ the dual maps are given by

$$\tilde{\pi}_i^{\dagger}(F) = F \setminus \{i\}$$

and

$$\tilde{\pi}_i(F) = \bigcup_{j \in F} \tilde{\pi}_i(\{j\})$$

where

$$\tilde{\pi}_i(\{j\}) = \begin{cases} \emptyset & \text{if } j = i, \\ \{i, j\} & \text{if } j \in \{i-1, i+1\}, \\ \{j\} & \text{otherwise,} \end{cases}$$

The graphical representation

We can give a graphical construction of the dual process as well. Define Poisson processes $(\tilde{N}_i(t))_{t\geq 0}$ and $(\tilde{N}_i^{\dagger}(t))_{t\geq 0}$ with the same parameter as previously, and let $(\tilde{T}_{i,n})_{n\geq 0}$ and $(\tilde{T}^{\dagger}_{i,n})_{n\geq 0}$ be their respective jump times.

Consider the time-space diagram $\mathbb{Z} \times \mathbb{R}_+$ again, and do as for the original process, but reversing the arrows.

A path is said to be *dual-valid* if it satisfies the following constraints:

- it never cross a "δ" mark,
- when moving upward, we never cross the tip of an arrow.

The graphical representation of the dual process

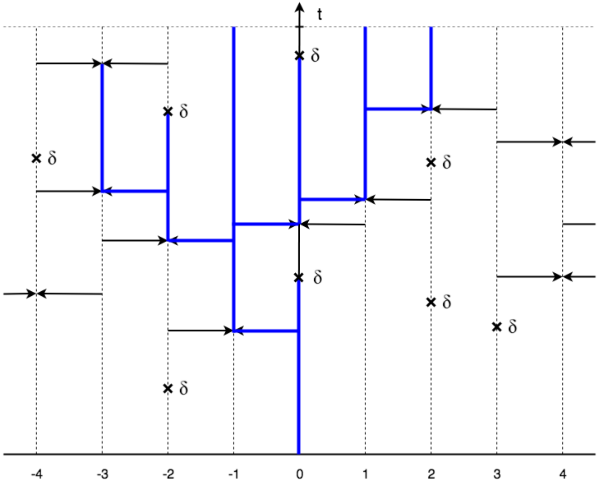

Figure: In blue all possible dual-valid paths starting from $(0,0)$ up to time t.

The duality relation

Then we can give an alternative characterization of the dual process as we did for the original process. For any $A \in \mathscr{P}_f(\mathbb{Z})$,

$$C^A(t) = \{j \in \mathbb{Z} : (i,0) \xrightarrow{\text{dual}} (j,t) \text{ for some } i \in A\}.$$

Proposition (Duality)

For any $B \in \mathscr{P}_f(\mathbb{Z})$, $A \in \mathscr{P}(\mathbb{Z})$, and $t \geq 0$ we have

$$\mathbb{P}\Big(\xi^A(t) \cap B \neq \emptyset\Big) = \mathbb{P}\Big(C^B(t) \cap A \neq \emptyset\Big).$$

Set monotonicity and stochastic monotonicity

Proposition (Set monotonicity)

Let $A \subset B \subset \mathbb{Z}$. Then for any $t \geq 0$ we have

$$\xi^A(t) \subset \xi^B(t).$$

Proposition (Stochastic monotonicity)

For any $0 \leq s < t$ we have the following

$$\mathbb{P}\left(\xi(s) \in \cdot\ \right) \geq \mathbb{P}\left(\xi(t) \in \cdot\ \right).$$

Upper-invariant and lower-invariant measures

Proposition

For any $\gamma > 0$ there exists a probability measures μ_γ which is invariant for $(\xi(t))_{t\geq 0}$ and that is such that

$$\mathbb{P}\Big(\xi(t) \in \cdot\ \Big) \underset{t\to\infty}{\longrightarrow} \mu_\gamma.$$

The Dirac measure on the "all zero" state, denoted $\delta_\emptyset$, is also invariant and we have

$$\mathbb{P}\Big(\xi^\emptyset(t) \in \cdot\ \Big) \underset{t\to\infty}{\longrightarrow} \delta_\emptyset.$$

Moreover if ν is any other invariant measure then $\delta_\emptyset \leq \nu \leq \mu_\gamma$.

Upper-invariant and lower-invariant measures for the dual

Proposition

For any $\gamma > 0$ there exists a probability measures $\tilde{\mu}_\gamma$ which is invariant for $(C(t))_{t\geq 0}$ and that is such that

$$\mathbb{P}\Big(C(t) \in \cdot\ \Big) \underset{t\to\infty}{\longrightarrow} \tilde{\mu}_\gamma.$$

The Dirac measure on the "all zero" state, denoted $\delta_\emptyset$, is also invariant and we have

$$\mathbb{P}\Big(C^\emptyset(t) \in \cdot\ \Big) \underset{t\to\infty}{\longrightarrow} \delta_\emptyset.$$

Moreover if ν is any other invariant measure then $\delta_\emptyset \leq \nu \leq \tilde{\mu}_\gamma$.

Upper-invariant measure in the sub-critical regime

Proposition

When $\gamma < \gamma_c$ *we have* $\rho_\gamma > 0$, *and therefore* $\mu_\gamma \neq \delta_\emptyset$.

Proposition

When $\gamma < \gamma_c$, *we have* $\tilde{\mu} \neq \delta_\emptyset$.

Proposition

In the sub-critical regime we have $\mu(\eta \equiv 0) = 0$ *and* $\tilde{\mu}(\eta \equiv 0) = 0$.

Space ergodicity of the Upper-invariant measure

Theorem

The measure μ_γ is spatially ergodic in the sense that a sequence of random variable $(X_k)_{k\in\mathbb{Z}}$ taking value in $\{0,1\}$ and such that X_k is distributed like $\mu_\gamma(\{\eta : \eta_k = \bullet\,\})$ would satisfy the following

$$\frac{1}{n+1}\sum_{k=0}^{n} X_k \xrightarrow[n\to\infty]{a.s.} \rho_\gamma .$$

Super-Critical Metastability

One dimensional lattice with nearest neighbour interaction and hard threshold

(work in progress)

Super-critical metastability

For $\gamma > \gamma_c$ we conjecture that the following convergence holds:

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N\to\infty]{\mathbb{P}} 1.$$

This follows immediately from the two following results (which remain to be proven):

$$\frac{\tau_N}{\log(N)} \xrightarrow[N\to\infty]{\mathbb{P}} C,$$

and

$$\frac{\mathbb{E}(\tau_N)}{\log(N)} \xrightarrow[N\to\infty]{} C.$$

Here C is a positive (and finite) constant.

Simulations

d-dimensional lattice with nearest neighbour interaction and hard threshold

Interaction structure

We did the simulation for $I = \mathbb{Z}$, $I = \mathbb{Z}^2$ and $I = \mathbb{Z}^3$. In each of these cases the structure of the interaction is nearest-neighbors.

- In the two-dimensional case the set of presynaptic neurons for neuron $i = (i_1, i_2)$ is

$$\mathbb{V}_i^2 = \{j = (j_1, j_2) \in \mathbb{Z}^2 : |i_1 - j_1| + |i_2 - j_2| = 1\}.$$

- In the three-dimensional case the set of presynaptic neurons for neuron $i = (i_1, i_2, i_3)$ is

$$\mathbb{V}_i^3 = \{j = (j_1, j_2, j_3) \in \mathbb{Z}^3 : |i_1 - j_1| + |i_2 - j_2| + |i_3 - j_3| = 1\}.$$

Interaction structure

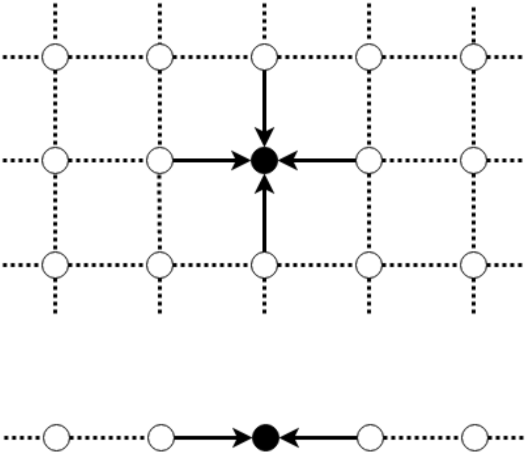

Figure: Interaction structure in dimension 1 and 2.

Histograms of the extinction time

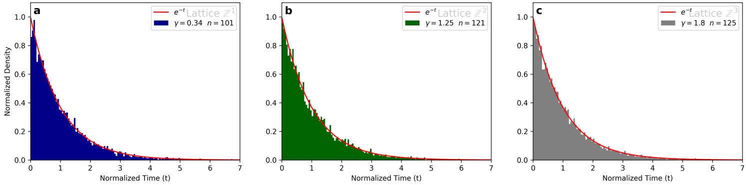

Figure: Sub-critical simulation for dimension 1, 2 and 3.

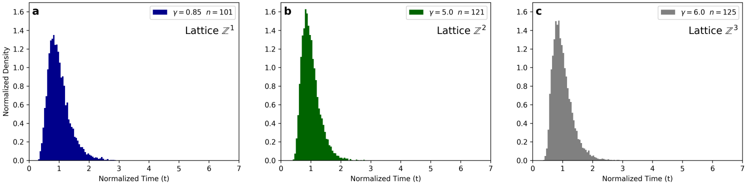

Figure: Super-critical simulation for dimension 1, 2 and 3.

Simulation for a varying number of neuron

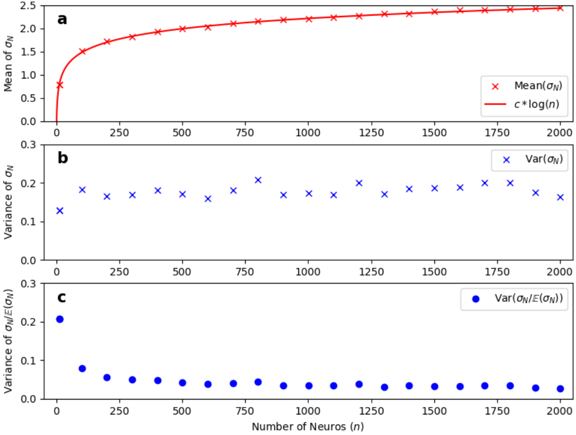

Figure: Fixed (super-critical) γ in the one-dimensional case for a varying number of neurons.

Open problems

Open problems

- In all developments we've evoked the rate function is chosen to be a hard threshold ($\phi_i(x) = \mathbb{1}_{x>0}$ for all $i \in I$). It makes the model more manageable mathematically but less realistic biologically. Could we prove the same results for a linear or a sigmoïd rate function?
- In the same lines, the nearest-neighbours structure of the interaction is mathematician-friendly (it allows us to use Harris graphical structure), but could we prove the same results for more complicated graphs (Erdős–Rényi for example)?

Simulation for Erdős–Rényi interaction graph

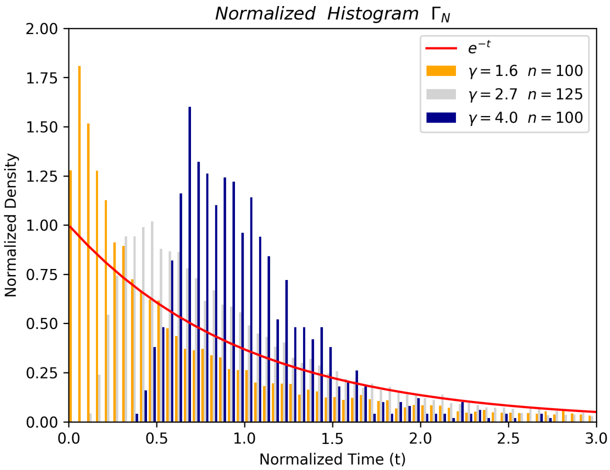

Figure: Simulation for Erdős–Rényi graph. The parameter of the Erdős–Rényi graph was chosen to be critical, i.e. $p_N = \frac{1}{N}$.

THE END

Thanks!