ICMNS2021

Morgan ANDRÉ

Metastability in Stochastic Systems of Spiking Neurons

13/11/2020

Overview

1. **What is Metastability?**

2. **The Model.**

3. **Result.**

4. **Other results and perspectives.**

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What is Metastability ?

What is Metastability (some quote)

> “Metastability is the property of a state, seemingly stable, but such that a tiny perturbation can push it toward an even more stable state.”

Wikipedia, Fr

The mathematician and physicist Bernard Derrida also used the expression **Dynamical phase transition**.

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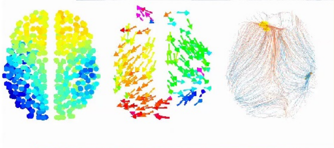
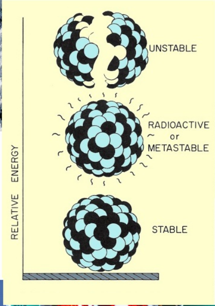
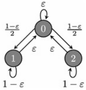
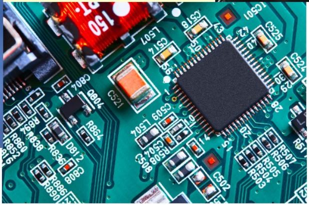
Some examples of metastable phenomenons





$$P = \begin{pmatrix} \varepsilon & \frac{1-\varepsilon}{2} & \frac{1-\varepsilon}{2} \\ \varepsilon & 1-\varepsilon & 0 \\ \varepsilon & 0 & 1-\varepsilon \end{pmatrix}$$





Some examples of metastable phenomenons

We can find a wide variety of examples from different field in which metastability arises:

- Physic: Supercooling water, avalanche, nuclear physics...
- Digital Electronics: Metastable bits.
- Mathematics: Catastrophe Theory in differential topology, Dynamical systems, Stochastic processes...
- Economics: Game Theory, Finance.
- Philosophy: Sartre, Simondon...

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Metastability in the brain

Metastability has been increasingly discussed in neuroscience during the last 20 years.

But often discussed in a loose sense, rarely from a mathematically rigorous perspective.

Some references:

- *"Metastability, criticality and phase transitions in brain and its models"*, G. Werner (2007).
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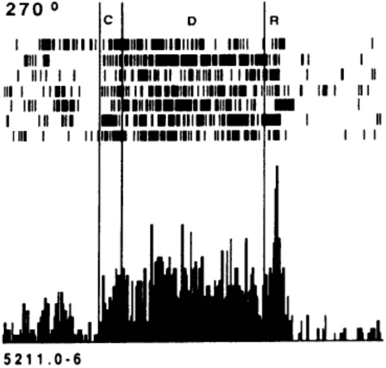
Metastability in the brain (some example?)



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Metastability in statistical mechanics

In 1984 Cassandro et al. introduced the following characterization in statistical mechanics for a metastable dynamic :

1. There is an absorbing state, but the time it takes to reach this state is exponentially distributed.

2. Before reaching this state the system behave like if it was described by the invariant measure of an other, closely related system.

They applied this approach to the Curie-Weiss model and to the Contact process.

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The Model

The Model

- A countable set S representing the **neurons**.
- For each neuron $i \in S$, a set $\mathbb{V}_i \subset I$ of **presynaptic neurons**.
- For each $i \in S$, a process $\left(X_i(t)\right)_{t \geq 0}$ taking value in $\mathbb{N}$ representing the **membrane potential** of neuron i.
- For each $i \in S$, two point processes $\left(N_i^*(t)\right)_{t \geq 0}$ and $\left(N_i^\dagger(t)\right)_{t \geq 0}$ representing **spiking times** and **total leak times** respectively.
- For each $i \in S$, a **spiking rate function** ϕ_i on $\mathbb{N}$.

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The point process $\left(N_i^{\dagger}(t)\right)_{t\geq 0}$ is a Poisson process of some rate $\gamma \geq 0$.

The point process $\left(N_i^{*}(t)\right)_{t\geq 0}$ has a fluctuating rate, given at time t by $\phi_i\big(X_i(t)\big)$.

The membrane potential at time t for neuron i is given by

$$X_i(t) = \sum_{j\in\mathbb{V}_i} \int_{]L_i(t),t[} dN_j^{*}(s),$$

and $L_i(t) = \sup\Big\{s \leq t : N_i^{*}(\{s\}) + N_i^{\dagger}(\{s\}) = 1\Big\}$.

The (simplified) Model

For all $i \in S$, $\phi_i(x) = \mathbb{1}_{x>0}$.

The (simplified) Model

We define our main object, denoted $\big(\xi(t)\big)_{t\geq 0}$, as follows

$$\forall t \geq 0,\ \forall i \in S, \quad \xi_i(t) \stackrel{\text{def}}{=} \mathbb{1}_{X_i(t)>0}.$$

This process is an **interacting particle system**. It is a continuous time Markov process taking value in $\{0,1\}^S$.

Depending on whether $\xi_i(t)$ is equal to 1 or 0 we will say that neuron i is **active** or **quiescent** respectively.

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One-dimensional lattice with nearest-neighbours interaction

Then we let $\left(\xi(t)\right)_{t\geq 0}$ be the process be defined on the one-dimensional lattice with nearest neighbours interaction, that is:

$$S = \mathbb{Z} \text{ and } \mathbb{V}_i = \{i-1, i+1\} \text{ for all } i \in S.$$

Let $\left(\xi_N(t)\right)_{t\geq 0}$ be the finite version of this process, that is the process defined on $S = [\![-N, N]\!]$ with

$$\mathbb{V}_i = \begin{cases} \{i-1, i+1\} & \text{if } i \in [\![-(N-1), N-1]\!], \\ \{N-1\} & \text{if } i = N, \\ \{-(N-1)\} & \text{if } i = -N. \end{cases}$$

One-dimensional lattice with nearest-neighbours interaction

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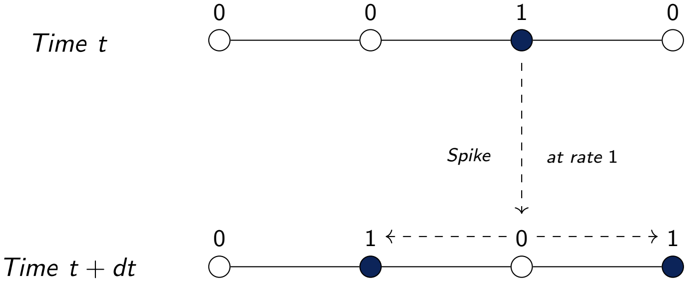
$\mathbb{Z}$ with nearest-neighbours interaction



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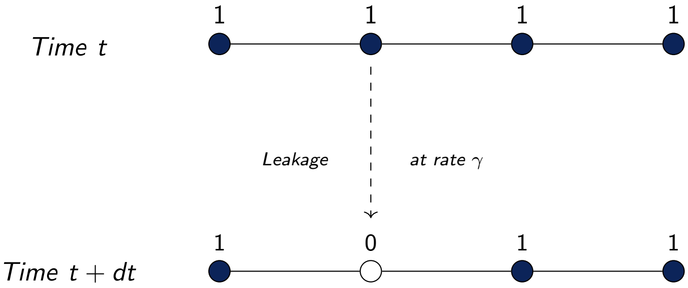
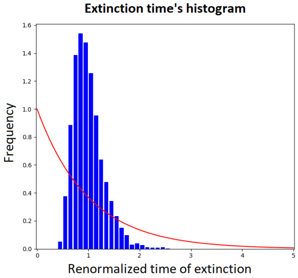
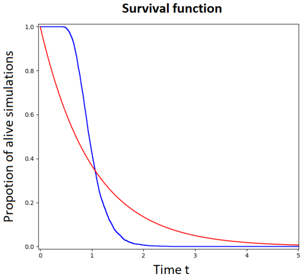
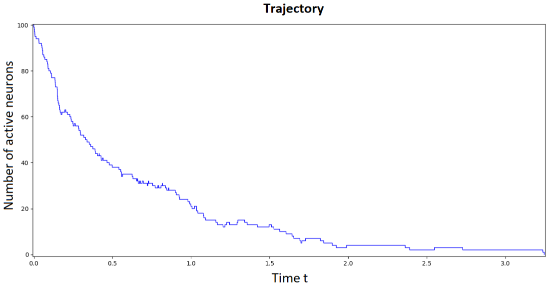


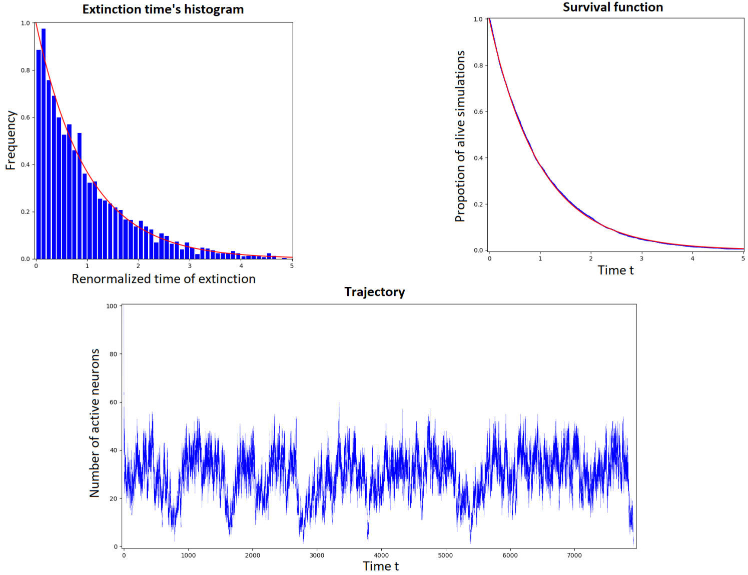
Simulations on the lattice for high γ.







Simulations on the lattice for low γ.



Result

Theorem

Define τ_N to be the time of extinction of $\left(\xi_N(t)\right)_{t\geq 0}$, that is:

$$\tau_N \overset{\text{def}}{=} \inf\{t \geq 0 : \xi_N(t)_i = 0 \text{ for any } i \in [\![-N, N]\!]\}.$$

Theorem

There exists γ_c such that if $0 < \gamma < \gamma_c$, then

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{E}(1).$$

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Proof for γ small

Proof.

Let β_N be the unique value in $\mathbb{R}_+$ such that

$$\mathbb{P}\left(\tau_N > \beta_N\right) = e^{-1}.$$

- First we show

$$\lim_{N\to\infty}\left|\mathbb{P}\left(\frac{\tau_N}{\beta_N} > s+t\right) - \mathbb{P}\left(\frac{\tau_N}{\beta_N} > s\right)\mathbb{P}\left(\frac{\tau_N}{\beta_N} > t\right)\right| = 0,$$

- from what we get

$$\frac{\tau_N}{\beta_N} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{E}(1),$$

- and we conclude with

$$\mathbb{E}\left(\tau_N\right) \underset{N\infty}{\sim} \beta_N.$$

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Proof of the first step

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We can prove that the infinite processes has nice properties. First it converges in the sense that there exists a measure μ such that

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We can therefore define the **asymptotical density** of the infinite process:

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A possibility in the case of the lattice is to consider the temporal means and to show that for any suitable function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, and for big R and N the following holds

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