Asymptotically Deterministic Time of Extinction for a Stochastic System of Spiking Neurons

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Description of the model

The model we are interested in is composed of:

- A countable set I representing the *neurons*.
- For each neuron $i \in I$, a set $\mathbb{V}_i \subset I$ of *presynaptic neurons*.
- For each $i \in I$, two point processes $\left(N_i^*(t)\right)_{t \geq 0}$ and $\left(N_i^\dagger(t)\right)_{t \geq 0}$ representing *spiking times* and *total leak times* respectively.
- For each $i \in I$, a real-valued process $\left(X_i(t)\right)_{t \geq 0}$ representing the *membrane potential* of neuron i.

Spiking times, leaking times and membrane potential

The point process $\left(N_i^\dagger(t)\right)_{t\geq 0}$ is a Poisson process of some rate $\gamma \geq 0$.

The point process $\left(N_i^*(t)\right)_{t\geq 0}$ is characterized by the property that for any $s \leq t$

$$\mathbb{E}\Big(N_i^*(t) - N_i^*(s)|\mathscr{F}_s\Big) = \int_s^t \mathbb{E}\Big(\phi_i(X_i(u))|\mathscr{F}_s\Big)du,$$

Where

$$X_i(t) = \sum_{j\in\mathbb{V}_i} \int_{]L_i(t),t[} dN_j^*(s),$$

and $L_i(t) = \sup\Big\{s \leq t : N_i^*(\{s\}) + N_i^\dagger(\{s\}) = 1\Big\}$.

$(\phi_i)_{i\in I}$ is a collection of rate functions.

One dimensional lattice with nearest neighbour interaction and hard threshold

Spiking times, leaking times and membrane potential

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Where

$$X_i(t) = \sum_{j\in\mathbb{V}_i} \int_{]L_i(t),t[} dN_j^*(s),$$

and $L_i(t) = \sup\left\{s \leq t : N_i^*(\{s\}) + N_i^\dagger(\{s\}) = 1\right\}$.

Spiking times, leaking times and membrane potential

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$$\mathbb{E}\Big(N_i^*(t) - N_i^*(s) | \mathscr{F}_s\Big) = \int_s^t \mathbb{E}\Big(\mathbb{1}_{X_i(u)>0} | \mathscr{F}_s\Big) du,$$

Where

$$X_i(t) = \int_{]L_i(t),t[} dN_{i-1}^*(s) + \int_{]L_i(t),t[} dN_{i+1}^*(s),$$

and $L_i(t) = \sup\Big\{s \leq t : N_i^*(\{s\}) + N_i^{\dagger}(\{s\}) = 1\Big\}$.

Phase transition

Theorem 1 (P. Ferrari et al.)

Suppose that for any $i \in \mathbb{Z}$ *we have* $X_i(0) \geq 1$*. There exists a critical value* γ_c *for the parameter* γ*, with* $0 < \gamma_c < \infty$*, such that for any* $i \in \mathbb{Z}$

$$\mathbb{P}\Big(N_i^*([0,\infty[) \; < \infty\Big) = 1 \qquad \textit{if } \gamma > \gamma_c,$$

and

$$\mathbb{P}\Big(N_i^*([0,\infty[) \; = \infty\Big) > 0 \qquad \textit{if } \gamma < \gamma_c.$$

Result of Metastability

For some $N \in \mathbb{Z}^+$ we set $I_N = \mathbb{Z} \cap [-N, N]$, and we write $\left(X_i^N(t)\right)_{t\geq 0}$ for the membrane potential of neuron i in the version of the model defined on the finite set I_N.

We define the *extinction time* of this model

$$\tau_N = \inf\left\{t \geq 0 : X_i^N(t) = 0 \text{ for all } i \in \mathbb{Z} \cap [-N, N]\right\}.$$

Theorem 2 (M. André)

If $\gamma < \gamma_c$, then we have the following convergence

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N\to\infty]{\mathscr{L}} \mathscr{E}(1).$$

Active and quiescent

The choice of $\mathbb{1}_{x>0}$ as the rate function has the important consequence that any given neuron can essentially be in only two states at any given time.

- When $X_i(t) = 0$ (or equivalently when $N_i^*(t)$ has rate 0), we say that neuron i is *quiescent*.
- When $X_i(t) \geq 1$ (or equivalently when $N_i^*(t)$ has rate 1), we say that neuron i is *active*.

The spiking rate process

We consider an important auxiliary process, namely the *spiking rates process*. Denoted $\big(\xi(t)\big)_{t\geq 0}$ and defined as follows

$$\forall t \geq 0,\ \forall i \in \mathbb{Z}, \qquad \xi_i(t) = \mathbb{1}_{X_i(t)>0}.$$

This process is an *interacting particle system*. It is a continuous time Markov process taking value in $\{0,1\}^{\mathbb{Z}}$. Each possible state is a doubly infinite sequence of 0 and 1, indicating in which state (quiescent or active) each neuron is.

The infinitesimal generator

The generator of the process $\big(\xi(t)\big)_{t\geq 0}$ is given by

$$\mathscr{L}f(\eta) = \gamma \sum_{i\in\mathbb{Z}} \Big(f(\pi_i^\dagger(\eta)) - f(\eta) \Big) + \sum_{i\in\mathbb{Z}} \eta_i \Big(f(\pi_i(\eta)) - f(\eta) \Big),$$

where the maps are given by

$$\pi_i^\dagger(\eta)_j = \begin{cases} 0 & \text{if } j = i, \\ \eta_j & \text{otherwise,} \end{cases}$$

and

$$\pi_i(\eta)_j = \begin{cases} 0 & \text{if } j = i, \\ \max(\eta_i, \eta_j) & \text{if } j \in \{i-1, i+1\}, \\ \eta_j & \text{otherwise.} \end{cases}$$

What we want to prove

Theorem 3

Suppose that $\gamma > \gamma_c$. Then the following convergence holds

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N\to\infty]{\mathbb{P}} 1.$$

What we want to prove

Theorem 4

Suppose that $\gamma > 1$. Then the following convergence holds

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N\to\infty]{\mathbb{P}} 1.$$

How we prove it

Proposition 5

Suppose that $\gamma > 1$*. Then there exists a constant* $0 < C < \infty$ *depending on* γ *such that the following convergence holds*

$$\frac{\tau_N}{\log(2N+1)} \xrightarrow[N\to\infty]{\mathbb{P}} C.$$

Proposition 6

Suppose that $\gamma > 1$*. Then the following convergence holds*

$$\frac{\mathbb{E}\left(\tau_N\right)}{\log(2N+1)} \xrightarrow[N\to\infty]{} C,$$

where C *is the same constant as in the previous proposition.*

Proof of Proposition 5

Notice that we have the following

$$\mathbb{P}\left(\xi^0(t+s) \neq \emptyset \mid \xi^0(t) \neq \emptyset\right) \geq \mathbb{P}\left(\xi^0(s) \neq \emptyset\right).$$

Which can be rewritten

$$\mathbb{P}\left(\xi^0(t+s) \neq \emptyset\right) \geq \mathbb{P}\left(\xi^0(t) \neq \emptyset\right) \mathbb{P}\left(\xi^0(s) \neq \emptyset\right).$$

Writing $f(t) = \log\left(\mathbb{P}\left(\xi^0(t) \neq \emptyset\right)\right)$, the previous line implies that f is superadditive. Therefore, by a well-known result (Fekete lemma), we have

$$\frac{f(t)}{t} \underset{t\to\infty}{\longrightarrow} -C', \tag{1}$$

where $C' = -\sup_{s>0} \frac{f(s)}{s}$.

Proof of Proposition 5

Notice that we also have for any $t > 0$

$$\mathbb{P}\left(\xi^0(t) \neq \emptyset\right) \leq e^{-C't}. \tag{2}$$

It is crucial to ensure that $0 < C' < \infty$. The $C' < \infty$ part is immediate from the inequality above but the inequality $C' > 0$ requires a little bit of work.

Proof of Proposition 5

Let $(Z_t)_{t\geq 0}$ denote a continuous time branching process with birth rate 1 and death rate γ.

We can realize a coupling between $(Z_t)_{t\geq 0}$ and our process $(\xi^0(t))_{t\geq 0}$ in such a way that $|\xi_t^0| \leq Z_t$ for any $t \geq 0$.

Then it follows that

$$\mathbb{P}\left(\xi^0(t) \neq \emptyset\right) \leq \mathbb{P}\left(Z_t \geq 1\right) \leq \mathbb{E}(Z_t) = e^{-(\gamma-1)t}.$$

This last inequality implies that $C' \geq \gamma - 1$, so that $C' > 0$ whenever $\gamma > 1$.

Proof of Proposition 5

We're aimed to prove that for any $\epsilon > 0$ the following holds

$$\mathbb{P}\left(\frac{\tau_N}{\log(2N+1)} - \frac{1}{C'} > \epsilon\right) \underset{N\to\infty}{\longrightarrow} 0, \tag{3}$$

and

$$\mathbb{P}\left(\frac{\tau_N}{\log(2N+1)} - \frac{1}{C'} < -\epsilon\right) \underset{N\to\infty}{\longrightarrow} 0. \tag{4}$$

Proof of Proposition 5 : equation (3)

Let's do the easy part! We have

$$\mathbb{P}\left(\xi_N(t) \neq \emptyset\right) \leq (2N+1)\mathbb{P}\left(\xi^0(t) \neq \emptyset\right) \leq (2N+1)e^{-C't}. \qquad (5)$$

Now take $t = (\frac{1}{C'} + \epsilon)\log(2N+1)$ and you get

$$\mathbb{P}\left(\frac{\tau_N}{\log(2N+1)} - \frac{1}{C'} > \epsilon\right) = P\left(\xi_N(t) \neq \emptyset\right) \leq e^{-C'\epsilon \log(2N+1)}.$$

It goes to 0 when N diverges as $C' > 0$.

Proof of Proposition 5 (equation 4)

Now the not so easy part. For some constant K to be fixed later and for any $k \in \mathbb{Z}$, we define:

$$F_k \;=\; \big\{(2k-1)K\log(2N+1), \ldots, (2k+1)K\log(2N+1)\big\}.$$

We also define the set of integers

$$I_N = \mathbb{Z} \cap \left[-\frac{N}{2K\log(2N+1)}, \frac{N}{2K\log(2N+1)}\right],$$

and the following configuration

$$A_N \;= \{2kK\log(2N+1) \text{ for } k \in I_N\}.$$

Proof of Proposition 5 (equation 4)

We then consider a modification of the process $(\xi_N(t))_{t\geq 0}$ where all neurons at the border of one of the sub-windows F_k defined above are fixed in quiescent state. This modified process is denoted $(\zeta_N(t))_{t\geq 0}$.

For any fixed time $t>0$, we define the following event

$$E_t = \Big\{ (\xi_s^0)_{0\leq s\leq t} \text{ stays inside } \{-K\log(2N+1),\ldots,K\log(2N+1)\} \Big\}.$$

Proof of Proposition 5 (equation 4)

Now for N big enough and for any $t > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left(\xi_N(t) = \emptyset\right) \\
&\leq \mathbb{P}\left(\zeta_N^{A_N}(t) = \emptyset\right) \\
&= \mathbb{P}\left(\zeta_N^0(t) = \emptyset\right)^{(2N+1)/(2K\log(2N+1))} \\
&\leq \left(\mathbb{P}\left(\zeta_N^0(t) = \emptyset \cap E_t\right) + \mathbb{P}\left(E_t^c\right)\right)^{(2N+1)/(2K\log(2N+1))} \\
&\leq \left(\mathbb{P}\left(\xi^0(t) = \emptyset\right) + \mathbb{P}\left(E_t^c\right)\right)^{(2N+1)/(2K\log(2N+1))}.
\end{aligned}$$

Proof of Proposition 5 (equation 4)

Now it only remains to find a suitable bound for $\mathbb{P}\left(\xi^0(t) = \emptyset\right)$ and for $\mathbb{P}\left(E_t^c\right)$.

For $\mathbb{P}\left(\xi^0(t) = \emptyset\right)$, we take write $\epsilon' = C'\epsilon$, and notice that $\frac{f(t)}{t} \geq -(1+\epsilon')C$ for big enough t, which can be written

$$\mathbb{P}\left(\xi_t^0 = \emptyset\right) \leq 1 - e^{-(1+\epsilon')C't}.$$

Now take $t = \left(\frac{1}{C'} - \epsilon\right)\log(2N+1) = \frac{1}{C'}\left(1-\epsilon'\right)\log(2N+1)$, to get

$$\mathbb{P}\left(\xi_t^0 = \emptyset\right) \leq 1 - \frac{1}{(2N+1)^{1-\epsilon'^2}}. \tag{6}$$

Proof of Proposition 5 (equation 4)

For $\mathbb{P}\left(E_t^c\right)$, we denote $r_t = \max \xi_t^0$ and we let $(M(t))_{t\geq 0}$ be an homogeneous Poisson process of parameter 1. We have for any $m \geq 0$

$$\mathbb{P}\left(\sup_{s\leq t} r_s \geq m\right) \leq \mathbb{P}\Big(M(t) \geq m\Big).$$

Moreover $\mathbb{E}\left(e^{M(t)}\right) = e^{t(e-1)}$, so by Markov inequality

$$\begin{aligned}\mathbb{P}\left(\sup_{s\leq t} r_s \geq K't\right) &\leq e^{t(e-1-K')} \\ &\leq e^{t(2-K')},\end{aligned}$$

Proof of Proposition 5 (equation 4)

Now taking again $t = \frac{1}{C'}(1-\epsilon')\log(2N+1)$ and $K' = 2(1+C')$ we get

$$\mathbb{P}\left(\sup_{s\leq t} r_s \geq m\right) \leq e^{-2(1-\epsilon')\log(2N+1)},$$

and assuming without loss of generality that $\epsilon' < \frac{1}{2}$ we get

$$\mathbb{P}\left(\sup_{s\leq t} r_s \geq m\right) \leq \frac{1}{2N+1}. \tag{7}$$

Proof of Proposition 5 (equation 4)

It is now possible to fix the value of the constant K:

$$K = \inf\left\{x \in \mathbb{R} \text{ such that } x \geq \frac{K'}{C'} \text{ and } x\log(2N+1) \in \mathbb{N}\right\}.$$

With these bounds in mind we get

$$\begin{aligned}
&\mathbb{P}\left(\xi_N(t) = \emptyset\right) \\
&\leq \left(\mathbb{P}\left(\xi^0(t) = \emptyset\right) + \mathbb{P}\left(E_t^c\right)\right)^{(2N+1)/(2K\log(2N+1))} \\
&\leq \left(1 - \left(\frac{1}{(2N+1)^{1-\epsilon'/2}} - \frac{2}{2N+1}\right)\right)^{(2N+1)/(2K\log(2N+1))}.
\end{aligned}$$

And this is easily proven to go to 0 when N goes to ∞.

Proof of Proposition 6

It remains to prove that for $\gamma > 1$ we also have

$$\frac{\mathbb{E}\left(\tau_N\right)}{\log(2N+1)} \underset{N\to\infty}{\longrightarrow} C.$$

It is well-known that the fact that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in probability to some random variable X doesn't necessarily implies that $\mathbb{E}(X_n) \underset{N\to\infty}{\longrightarrow} \mathbb{E}(X)$.

Nonetheless this implication holds true with the additional assumption that the sequence is uniformly integrable, i.e. if

$$\lim_{M\to\infty}\left(\sup_{n\in\mathbb{N}} \mathbb{E}\Big(|X_n|\mathbb{1}_{\{|X_n|>M\}}\Big)\right) = 0.$$

Proof of Proposition 6

We have the following

$$\mathbb{E}\left(\frac{\tau_N}{\log(2N+1)}\mathbb{1}_{\{\frac{\tau_N}{\log(2N+1)}>M\}}\right)$$
$$=\int_0^{\infty}\mathbb{P}\left(\frac{\tau_N}{\log(2N+1)}>\max(t,M)\right)dt,$$

which leads to

$$\sup_{n\in\mathbb{N}^*}\mathbb{E}\left(\frac{\tau_N}{\log(2N+1)}\mathbb{1}_{\{\frac{\tau_N}{\log(2N+1)}>M\}}\right)\leq 3^{1-C'M}\left[M+\frac{1}{C'\log(3)}\right].$$

THE END

Thanks!