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FUNDAÇÃO DE AMPARO À PESQUISA DO ESTADO DE SÃO PAULO

Temporal Averages of a Metastable System of Interacting Point-processes Representing Spiking Neurons

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1 Definition of the model

Let S be a finite or countable set representing the neurons, and to each $i \in S$ associate a set $\mathbb{V}_i \subset S$ of *presynaptic neurons*. Each neuron is associated with three processes: a Poisson process $(N_i^\dagger(t))_{t\geq 0}$ of parameter γ, representing the *leak times*, another point process $(N_i(t))_{t\geq 0}$ representing the *spiking times*, and a process $(X_i(t))_{t\geq 0}$ which taking values in the set $\mathbb{N}$ of non-negative integers and which represents the *membrane potential*. The spiking activity also depends on a *rate function* $\phi : \mathbb{N} \mapsto \mathbb{R}_+$. The dynamic of the system is completely characterized by the following set of equations

$$\mathbb{E}(N_i(t) - N_i(s)|\mathcal{F}_s) = \int_s^t \mathbb{E}(\phi(X_i(u))|\mathcal{F}_s)du \tag{1}$$

where

$$X_i(t) = \sum_{j\in\mathbb{V}_i} \int_{]L_i(t),t[} dN_j(s), \tag{2}$$

$L_i(t)$ being the time of the last event affecting neuron i before time t, that is,

$$L_i(t) = \sup\Big\{s \leq t : N_i(\{s\}) = 1 \text{ or } N_i^\dagger(\{s\}) = 1\Big\}.$$

$(\mathcal{F}_t)_{t\geq 0}$ is the standard filtration for the point processes involved here, that is the filtration which at any time $t \geq 0$ is equal to the σ-algebra generated by the family $\{N_i(s), N_i^\dagger(s), s \leq t, i \in S\}$.

2 An auxiliary interacting particle system

The model above is then considered in the specific case where ϕ is defined by $\phi : x \mapsto \phi(x) = \mathbb{1}_{x>0}$ (as in [FER+18]). Now for any $i \in S$ and $t \geq 0$ we write $\eta_i(t) = \mathbb{1}_{X_i(t)>0}$, and $\eta(t) = (\eta_j(t))_{j\in S}$, then the resulting process $(\eta(t))_{t\geq 0}$ is an **interacting particle system**, that is a Markovian process taking value in $\{0,1\}^S$. It has the following infinitesimal generator:

$$f(\xi) = \gamma \sum_{i\in S} \Big(f(\pi_i^\dagger(\xi)) - f(\xi)\Big) + \sum_{i\in S} \xi_i \Big(f(\pi_i(\xi)) - f(\xi)\Big),$$

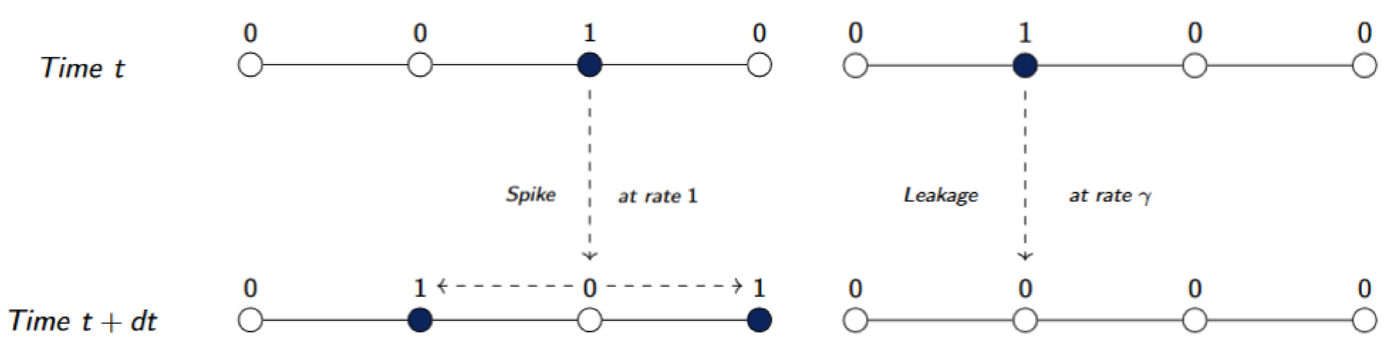
where the maps are given by

$$\pi_i^\dagger(\xi)_j = \begin{cases} 0 & \text{if } j = i, \\ \xi_j & \text{otherwise,} \end{cases}$$

and

$$\pi_i(\xi)_j = \begin{cases} 0 & \text{if } j = i, \\ \max(\xi_i, \xi_j) & \text{if } i \in \mathbb{V}_j, \\ \xi_j & \text{otherwise.} \end{cases}$$

In the case of the one-dimensional lattice with nearest neighbors interaction version of the system — that is $S = \mathbb{Z}$ or $S = [\![-n, n]\!]$ and $\mathbb{V}_i = \{i-1, i+1\}$ — the dynamic looks as follows.



3 Metastability

The phenomenon of **metastability** has raised a lot of interest during the last decades in the neuroscientific community and it is considered by many to be intrinsically linked to the brain ability to process information in a coordinated manner in a noisy, constantly changing environment (a neuroscientific survey on the subject can be found in [WER07]). The concept doesn't come from neuroscience though, and it has now a long history in physics and in the theory of stochastic processes, in which it designates the behavior of a system which evolves in a pseudo-stationary manner for an unpredictably long time, before falling into the actual equilibrium because of an infinitesimally rare but macroscopically unavoidable deviation from this pseudo-stationary phase. A precise characterization was proposed in [CAS+84], consisting in the two following conditions:

(i) the time it takes to exit the pseudo-stationary phase converges to a **memory-less** distribution (i.e. an exponential distribution) when the number of components in the system goes to infinity;

(ii) and before the exit time the behavior of the system behave in a way that **resemble stationarity** in the sense that temporal averages are close to some predetermined theoretical value.

3.1 Asymptotic memorylessness of the time of extinction

The first of these two points was obtained in [AND19]. The object of interest is the last spiking time of the system, that is:

$$\tau_n = \inf\{t \geq 0 : N_i([t, \infty[) = 0 \text{ for all } i \in S\}. \tag{3}$$

If S is finite then the extinction time above is necessarily finite. Moreover, it can be shown that

Theorem 3.1. *Suppose that the family of point processes $\big((N_i(t))_{t\geq 0}, i \in S\big)$ satisfies the equations (1) and (2) for $S = [\![-n, n]\!]$ and $\mathbb{V}_i = \{i-1, i+1\} \cap S$. Moreover let τ_n be defined as in (3). Then, there exists a critical value $\gamma_c > 0$ such that we have the following convergence whenever $0 < \gamma < \gamma_c$:*

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{E}(1).$$

This result is obtained by an in-depth analysis of the infinite version of the auxiliary interacting particle system. In particular important properties of this IPS are obtained by studying its dual. Then the asymptotic memorylessness is obtained via path-wise approach inspired by [CAS+84].

3.2 A pseudo-stationary phase

The second characteristic property of metastability was investigated in [AND22]. It was proven that for the finite version of the system, in the sub-critical regime and if n is big enough, then counting the number of spikes occurring in a given time interval before extinction, for a given subset of neurons of interest in the system, shall give a number which with high probability cannot be too far from some fixed value. In other words, before extinction the system essentially keeps emitting spikes at a constant rate.

More precisely, let $F \subset \mathbb{Z}$ with $|F| < \infty$, and for any $n \in \mathbb{N}$ let $F_n = F \cap [\![-n, n]\!]$. Then we define, for any $t, R \in \mathbb{R}_+$ and for any $n \geq 0$, the following quantity

$$\widehat{N}_R^n(t, F) = \frac{1}{R} \sum_{i\in F_n} N_i([t, t+R]). \tag{4}$$

$\widehat{N}_R^n(t, F)$ is the average number of spikes emitted by the neurons in F_n on a time interval of length R, starting the enumeration a time t. The following result holds.

Theorem 3.2. *Suppose that the family of point processes $\big((N_i(t))_{t\geq 0}, i \in S\big)$ satisfies the equations (1) and (2) for $S = [\![-n, n]\!]$ and $\mathbb{V}_i = \{i-1, i+1\} \cap S$ for any $i \in S$. Moreover let τ_n be defined as in (3) and, for any $n \in \mathbb{N}$, $R > 0$ and $t \geq 0$, let also $\widehat{N}_R^n(t, F)$ be defined as in (4). Finally let $0 < \gamma < \gamma_c$ and let $(R_n)_{n\geq 0}$ be an increasing sequence of positive real numbers satisfying*

$$R_n \xrightarrow[n\to\infty]{} +\infty \quad \textit{and} \quad \frac{R_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{} 0.$$

Then there exists some $0 < \rho < 1$ (which depends only on γ) such that for any $t \geq 0$

$$\widehat{N}_{R_n}^n(t, F) \xrightarrow[n\to\infty]{\mathbb{P}} |F| \cdot \rho.$$

This result is obtained by studying in parallel both the system of interacting point processes and its auxiliary IPS. In particular $\rho = \lim_{t\to\infty} \mathbb{P}(\eta_0(t) = 1)$ and we know that $\rho > 0$ whenever $\gamma < \gamma_c$. One of the most important intermediary result on the way to proving this theorem is the following proposition.

Proposition 3.3. *Let $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ be a function depending on finitely many coordinates. There exists positive constants C_1 and C_2 (depending on f) such that for any $s, t \in \mathbb{R}^+$*

$$|\mathrm{Cov}(f(\eta(s)), f(\eta(t)))| \leq C_1 e^{-C_2|s-t|}.$$

In words, the auxiliary IPS has exponentially decaying time correlations. This result is again obtained by a thorough investigation of its dual. Moreover, having fast decay of the time correlations implies — in a somewhat loose sense — that the system is producing a lot of entropy, so that intuitively it has no reason to go very far from some basal level activity.

References

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