Metastability for an infinite system of spiking neurons

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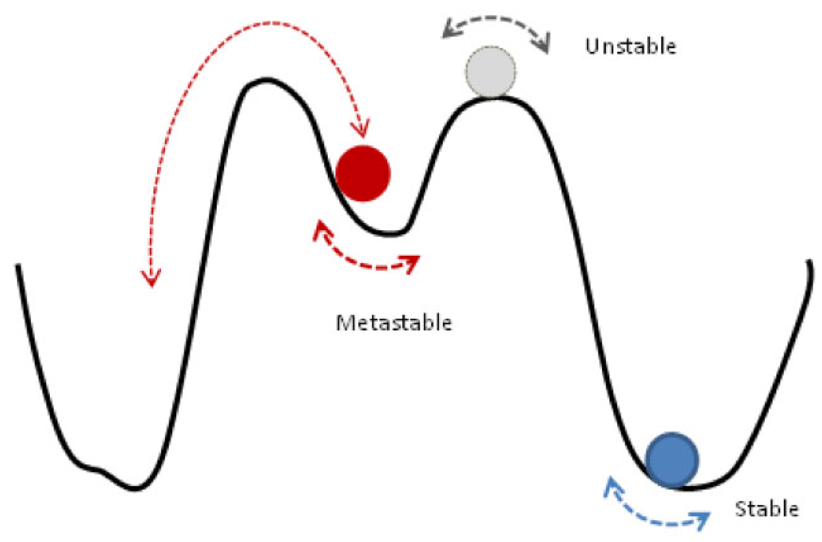
1 Introduction

In [1] was introduced a stochastic process for an infinite system of spiking neurons. This process is an additive *interacting particle system* and it is linked, in a way that is described here, to the famous *contact process*.

The contact process (when the starting set is chosen appropriately) has the noticeable property that it is *metastable*. Metastability is a notion that arises in various fields of science, specifically in physic and chemistry, and it denotes the tendency for a system to stay a long time in a state that presents the appearance of stability, but which is such that a small perturbation can (and will) suddenly lead it to a state that is actually of higher stability. Metastability for stochastic processes can be stated in a formal way, and, for the contact process, the proof of its metastability can be found in [2] and [3].

For reasons that are exposed here, we believe that the results obtained about the contact process can be obtained for the process presented in [1] aswell. Such a feature for this stochastic process is particularly appealing as it corroborates the widespread conjecture in the neuroscience community that the brain presents a highly metastable behavior.



2 Infinite system of spiking neurons

The system of neurons can be described as follows.

- I is a countable set (representing the neurons).
- We attach to all $i \in I$ two point processes $\big(N_i^{\dagger}(t)\big)_{t\geq 0}$ and $\big(N_i^{*}(t)\big)_{t\geq 0}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with standard filtration $\big(\mathcal{F}_t\big)_{t\geq 0}$.

$\big(N_i^{\dagger}(t)\big)_{t\geq 0}$ is a family of i.i.d. poisson point processes of rate $\gamma \geq 0$. They represent the *leak time* of the system of neurons.

$\big(N_i^{*}(t)\big)_{t\geq 0}$ is a family characterized by the property that, for all $s \leq t$,

$$\mathbb{E}\big(N_i^*(t) - N_i^*(s)|\mathcal{F}_s\big) = \int_s^t \mathbb{E}\big(\mathbf{1}_{X_i(u)\geq 0}|\mathcal{F}_s\big)du$$

where

$$X_i(t) = \sum_{j\in\{i-1,i+1\}} \int_{]L_i(t),t[} dN_j^*(s)$$

and

$$L_i(t) = \sup\{s \leq t : N_i^*(s) + N_i^{\dagger}(s) = 1\}$$

From a neurobiological point of view $\big(X_i(t)\big)_{t\geq 0}$ represents the *membrane potential* of neuron i and $(N_i^*(t))_{t\geq 0}$ are the *spiking times* of neuron i.

Moreover, for all $i \in I$ and $t \geq 0$ we denote $\eta_i(t) = \mathbf{1}_{X_i(t)>0}$ and $\eta(t) = \big(\eta_j(t), j \in I\big)$. $\big(\eta(t)\big)_{t\geq 0}$ is stochastic process taking values in $\{0,1\}^I$ and belongs to the class of *interacting particle system*.

3 Link with the contact process

As it has been said, the previous process shows some similarities with the contact process. The later can be defined informally as follows (in the one dimensional case): each site of $\mathbb{Z}$ is given a state 0 or 1, which can be seen as the state of an infection (1 for "infected", 0 for "healthy"), the spin of a particle, the state of an opinion etc. The dynamic is the following, at any site $x \in \mathbb{Z}$, if the site is infected, the infection will disappear ($1 \to 0$) at rate 1 ; if the site is healthy, it will be infected ($0 \to 1$) at rate $\lambda \cdot \#\{\text{infected neigbhours}\}$ for some parameter $\lambda > 0$.

The contact process notoriously presents a *phase transition* phenomenon. In other words, there exists a critical parameter λ_c such that the infection will die out for $\lambda < \lambda_c$ (which means that it will converge to the trivial invariant measure $\delta_{\emptyset}$), and such that there exists a non-trivial invariant measure to which the process will converge (in the sense of *weak convergence*) for $\lambda > \lambda_c$.

The first similarity between the contact process and the system of spiking neurons lies in the fact that the latter presents the same kind of phase transition property with respect to the parameter γ. This is the object of the following theorem, which can be found in [1].

Theorem 1. *Assuming that $X_i(0) \geq 1$ for all $i \in I$, there exists γ_c with $0 < \gamma_c < \infty$ such that he following holds. For all $i \in \mathbb{Z}$,*

$$\mathbb{P}\big(N_i^*([0,\infty[) < \infty\big) = 1, \text{ if } \gamma > \gamma_c$$

and

$$\mathbb{P}\big(N_i^*([0,\infty[) = \infty\big) > 0, \text{ if } \gamma < \gamma_c$$

Moreover, $\big(\eta_t\big)_{t\geq 0}$ has been proved to be an additive process in the sense of Harris (see [4]), and as a consequence it has a dual process, a pure jump Markov process that take values in $\mathcal{P}_f(\mathbb{Z})$ (the class of all subsets of $\mathbb{Z}$), which we denote $(C^A(t))_{t\geq 0}$ for any starting set A. We will say that A is simply connected if it satisfies the following definition.

Definition 1. A subset A in $\mathcal{P}_f(\mathbb{Z})$ is said to be *simply connected* if $|A| \geq 2$ and if for any k and j in A such that $j < k$ we have that $l \in A$ for all $j \leq l \leq k$. If A is not simply connected we will say it is *disconnected*.

Informally, being simply connected simply means that the set is a "one-block" set. We also define the following stopping time

$$\tau_A = \inf\{t > 0, C^A(t) \text{ is disconnected}\}.$$

Now, the second link between the contact process and the system of spiking neurons lies in the following proposition (that corresponds partly to lemma 3 in [1])

Proposition 1. *If A is simply connected, then the process $\big(C^A(t \bigwedge \tau_A)\big)_{t\geq 0}$ follows the dynamics of a classical contact process with infection rate 1 and with recovery rate γ.*

4 Metastability

As it has been said that the contact process presents a metastable behavior. To be more specific it is metastable in the super-critical case ($\lambda > \lambda_c$). Informally, this means the following:

- It stays ou of its equilibrium in a memoryless random time (theorem 1 in [2])
- During this time in which the system is out of equilibrium it stabilizes in the sense that it behave as if it was described by an other probability measure. This is sometimes referred as *thermalization*.
- Finally, and abruptly, it goes to the true equilibrium.

More formally, the first item of the previous list can be rigorously formulated as follows. If we denote by $(\xi_N(t))_{t\geq 0}$ the contact process restricted to the set $\{-N, -N+1, \ldots, N-1, N\}$ for some $N > 0$, starting from the simply connected set where everybody is infected, and if we define the time of extinction as

$$T_N = \textit{inf}\{t > 0 : \xi_N(t) = \emptyset\}.$$

Then then the following theorem holds, for some normalization constant β_N.

Theorem 2. *If $\lambda > \lambda_c$, then the random variable $\frac{T_N}{\beta_N}$ converges in the sense of weak convergence to a unit exponential random variable as $N \to \infty$.*

The exposed similarities between the contact process and the system of spiking neurons, and specifically the proposition 1, lead us to think that the latter theorem, as well as the formal results corresponding to the thermalization and the final abrupt convergence to equilibirum (for example theorems 2 and 3 in [2]) can be directly adapted and proved in the case of the system of spiking neurons.

References

[1] P.A. Ferrari, A. Galves, I. Grigorescu and E. Löcherbach *Phase transition for infinite systems of spiking neurons*, 2018.

[2] R. Schonmann *Metastability for the contact process*, Journal of Statistical Physics, 1985.

[3] M. Cassandro, A. Galves, E. Olivieri and M. Vares *Metastable behavior of stochastic dynamics: a pathwise approach*, Journal of Statistical Physics, 1984.

[4] T. E. Harris *On a class of set-valued Markov processes*, Annals of Probability, 1976.